

MULTIVARIATE REGULAR VARIATION OF DISCRETE MASS FUNCTIONS WITH APPLICATIONS TO PREFERENTIAL ATTACHMENT NETWORKS

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Abstract

Regular variation of a multivariate measure with a Lebesgue density implies the regular variation of its density provided the density satisfies some regularity conditions. Unlike the univariate case, the converse also requires regularity conditions. We extend these arguments to discrete mass functions and their associated measures using the concept that the mass function can be embedded in a continuous density function. We give two different conditions, monotonicity and convergence on the unit sphere, both of which can make the discrete function embeddable. Our results are then applied to the preferential attachment network model, and we conclude that the joint mass function of in- and out-degree is embeddable and thus regularly varying.

Keywords: Multivariate regular variation, preferential attachment, random graphs, power laws, in-degree, out-degree.

1 Introduction

The influence of heavy tailed modeling methods has spread to many fields. Application areas for the modeling and statistical methods include finance [27], insurance [12], social networks and random graphs [11, 4, 25, 26], mobility modeling for wireless phone users [15], parallel processing queueing models of cloud computing [14], models to optimize power usage when a mobile user changes between wifi and mobile networks [16].

The theory of regular variation is an essential mathematical tool in the analysis of heavy tailed phenomena. A measurable function f is regularly varying with index $\alpha > 0$ (written $f \in RV_\alpha$) if $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfies

$$\lim_{t \rightarrow \infty} \frac{f(tx)}{f(t)} = x^\alpha, \quad \text{for } x > 0.$$

In the univariate case, Karamata's theorem gives the asymptotic properties of the indefinite integral; differentiation of a regularly varying integral to recover the density function is covered by the monotone density theorem; see [23, Chapter 2.3] and [2, page 38]. Roughly, if the derivative of f is regularly varying with index α , then f is regularly varying with index $\alpha + 1$. Conversely, when f is monotone, the derivative of $f \in RV_\alpha$ is regularly varying with index $\alpha - 1$. These results can

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be interpreted as relating regular variation properties of an absolutely continuous measure and the density of the measure.

In practice, collected data are often multidimensional which raises the issue of generalizing the one-dimensional theory relating an integral and derivative to higher dimensions. Unfortunately additional regularity conditions are indispensable for the generalizations. This is discussed in a series of publications [8, 10, 9, 7, 19]. An additional issue is that in the era of Internet and social network studies, many data sets are discrete. For example, when studying the growth of social networks, the in- and out-degrees of nodes exhibit power laws. Under such circumstances, ignoring the discrete essence of the data is not appropriate and we need to understand the relationship between regular variation properties of a discretely supported measure and its mass function.

We proceed using the idea of *embedding* a function of a discrete variable in a function of a continuous variable. In one dimension ([3], [2, Theorem 1.9.5]), a regularly varying sequence can always be embedded in a regularly varying function of a continuous variable. The embeddability property in the multivariate case is not as obvious because the definition of regular variation exerts no control over the function's variation when we move from ray to ray and additional conditions are required for embeddability. Once this is resolved the results in the continuous cases can be used to examine the relationship between the regular variation of a mass function and its associated measure.

We apply our results to the preferential attachment network model studied in [4, 17, 26, 25, 28] where a new node attaches to an existing node or new edges are created according to probabilistic postulates that take into account the current in- and out-degrees of the existing nodes. The joint asymptotic distribution of in- and out-degree has multivariate regularly varying tails [25, 26]. We check the embeddability conditions for the joint mass function of in- and out-degree and conclude that the mass function is also regularly varying.

This paper is organized as follows. We start with a brief overview of the multivariate regular variation of both measures and functions in Section 2.1 and then state the univariate embeddability results in Section 2.2. We describe the preferential attachment model in Section 2.3. Sections 3 and 4 provide two different conditions to establish embeddability in the bivariate case and also characterize the relationship between the regular variation of a mass function and its measure. We then apply our results to the preferential attachment model in Sections 3.1 and 4.1, verifying that the joint mass function of in- and out-degrees is also regularly varying.

2 Preliminaries

2.1 Multivariate Regular Variation

We briefly review the basic concepts of multivariate regular variation for measures with emphasis on the two dimensional case. We use \mathbb{M} -convergence to define regular variation instead of the traditional way of using vague convergence. See [13, 5, 18] for the details on \mathbb{M} -convergence and reasons for its use.

Consider \mathbb{R}_+^2 metrized by a convenient metric $d(x, y)$. A subset $\mathbb{C} \subseteq \mathbb{R}_+^2$ is a cone if it is closed under positive scalar multiplication: if $\mathbf{v} \in \mathbb{C}$ then $c\mathbf{v} \in \mathbb{C}$ for $c > 0$. A proper framework for discussing regular variation is measure convergence on a closed cone $\mathbb{C} \subseteq \mathbb{R}_+^2$ with a closed cone $\mathbb{C}_0 \subseteq \mathbb{C}$ deleted. \mathbb{C}_0 is called the *forbidden zone*. In this paper we are interested in the case where $\mathbb{C} = \mathbb{R}_+^2$ and $\mathbb{C}_0 = \{\mathbf{0}\}$. Then $\mathbb{E} = \mathbb{R}_+^2 \setminus \{\mathbf{0}\}$ is the space for defining \mathbb{M} -convergence appropriate for regular variation of distributions of positive random vectors. The forbidden zone is the origin $\{\mathbf{0}\}$.

Let $\mathbb{M}(\mathbb{C} \setminus \mathbb{C}_0)$ be the set of Borel measures on $\mathbb{C} \setminus \mathbb{C}_0$ which are finite on sets bounded away from the forbidden zone \mathbb{C}_0 . We now give the definition of \mathbb{M} -convergence which becomes the basis

for our definition of multivariate regular variation of measures.

Definition 1. For $\mu_n, \mu \in \mathbb{M}(\mathbb{C} \setminus \mathbb{C}_0)$ we say $\mu_n \rightarrow \mu$ in $\mathbb{M}(\mathbb{C} \setminus \mathbb{C}_0)$ if $\int f d\mu_n \rightarrow \int f d\mu$ for all bounded, continuous, non-negative f on $\mathbb{C} \setminus \mathbb{C}_0$ whose support is bounded away from \mathbb{C}_0 .

A random vector $(X, Y) \geq 0$ is non-standard regularly varying on $\mathbb{C} \setminus \mathbb{C}_0$ if there exists $b_1(t) \in RV_{1/\alpha_1}$ and $b_2(t) \in RV_{1/\alpha_2}$ ($\alpha_1, \alpha_2 > 0$), called the scaling functions, and a measure $\nu(\cdot) \in \mathbb{M}(\mathbb{C} \setminus \mathbb{C}_0)$, called the limit or tail measure, such that as $t \rightarrow \infty$,

$$t\mathbf{P} \left[\left(\frac{X}{b_1(t)}, \frac{Y}{b_2(t)} \right) \in \cdot \right] \rightarrow \nu(\cdot), \quad \text{in } \mathbb{M}(\mathbb{C} \setminus \mathbb{C}_0).$$

When $b_1(t) = b_2(t)$, (X, Y) is said to have a distribution with *standard* regularly varying tails with index $\alpha := \alpha_1 = \alpha_2$ and the limiting measure ν satisfies the scaling property: $\nu(c \cdot) = c^{-\alpha} \nu(\cdot)$ for $c > 0$. Without loss of generality, we assume all scaling functions are continuous and strictly increasing.

Following the definition in [10], we say a measurable function $f : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ is *multivariate regularly varying with scaling functions b_1 and b_2 and limit function λ* , if there exists $h : (0, \infty) \mapsto (0, \infty)$ with $h \in RV_\alpha$ for some $\alpha \in \mathbb{R}$ such that

$$\lim_{t \rightarrow \infty} \frac{f(b_1(t)x, b_2(t)y)}{h(t)} = \lambda(x, y) > 0, \quad \forall x, y > 0. \quad (2.1)$$

If both b_1 and b_2 are the identity function, we get ordinary regular variation as in [8].

2.2 Regularly Varying Functions of Discrete Variables and Embeddability

Internet modeling and social network studies require many relevant variables to be discrete. A natural example is in- and out-degree of nodes in a random graph. So it is useful to examine regular variation for functions of discrete variables. For the one dimensional case, see [3, 2].

Definition 2. A sequence $(c_n)_{n \in \mathbb{N}}$ of positive numbers is *regularly varying with index $\alpha \in \mathbb{R}$* if

$$\lim_{n \rightarrow \infty} c_{[nx]}/c_n = x^\alpha > 0, \quad x > 0. \quad (2.2)$$

A doubly indexed function $k : \mathbb{Z}^2 \setminus \{\mathbf{0}\} \mapsto \mathbb{R}_+$ is *regularly varying with scaling functions b_1 and b_2 and limit function $\lambda(x, y)$* if for some $h \in RV_\alpha$ for some $\alpha \in \mathbb{R}$, $b_i \in RV_{\beta_i}$, $\beta_i > 0$, we have

$$\lim_{n \rightarrow \infty} \frac{k([b_1(n)x], [b_2(n)y])}{h(n)} = \lambda(x, y) > 0, \quad \forall x, y > 0. \quad (2.3)$$

In one dimension, a regularly varying sequence can always be embedded in a regularly varying function of a continuous argument [2, Theorem 1.9.5]:

Theorem 2.1. If (c_n) is regularly varying in the sense of (2.2), then the function

$$f(x) := c_{[x]}, \quad x > 0, \quad (2.4)$$

varies regularly with index α , where $[x]$ denotes the largest integer smaller than or equal to x .

So every regularly varying sequence is *embeddable* meaning that there exists a regularly varying function $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $c_n = f(n)$, and (2.4) implies $c_{n-1}/c_n \rightarrow 1$ as $n \rightarrow \infty$. To extend embeddability to the multivariate function $k(i, j)$, we need to make additional assumptions on the behavior of the function $k(i, j)$. Similar to (2.2), when k satisfies (2.3), we say $k(i, j)$ is *embeddable* if there exists a bivariate regularly varying function $g(x, y)$ satisfying (2.1) such that $g(x, y) := k([x], [y])$.

2.3 Preferential Attachment Network Models

The directed edge preferential attachment model [17, 4] is a model for a growing directed random graph. The model evolves according to certain rules. Choose strictly positive parameters $\alpha, \beta, \gamma, \lambda, \mu$ such that $\alpha + \beta + \gamma = 1$, and additionally assume that $\alpha, \beta, \gamma < 1$ to avoid trivial cases. The initial condition for the model is a finite directed graph, denoted by $G(n_0)$, with at least one node and n_0 edges. For $n = n_0 + 1, n_0 + 2, \dots$, $G(n)$ is a graph with n edges and a random number $N(n)$ of nodes. If a node v is from $G(n)$, use $D_{in}(v)$ and $D_{out}(v)$ to denote its in and out degree respectively (dependence on n is suppressed). Then $G(n+1)$ is obtained from $G(n)$ as follows.

- (i) With probability α a new node w is born and we add an edge leading from w to an existing node $v \in G(n)$ (written as $w \mapsto v$). The existing node v is chosen with probability according to its in-degree:

$$\mathbf{P}(v \in G(n) \text{ is chosen}) = \frac{D_{in}(v) + \lambda}{n + \lambda N(n)}.$$

- (ii) With probability β we add a directed edge $v \mapsto w$ between two existing nodes $v, w \in G(n)$. Nodes v and w are chosen independently from all the nodes of $G(n)$ with probabilities

$$\mathbf{P}(v \text{ is chosen}) = \frac{D_{out}(v) + \mu}{n + \mu N(n)}, \text{ and } \mathbf{P}(w \text{ is chosen}) = \frac{D_{in}(w) + \lambda}{n + \lambda N(n)}.$$

- (iii) With probability γ a new node w is born and we add an edge leading from an existing node $v \in G(n)$ to w . The existing node v is chosen with probability according to its out-degree:

$$\mathbf{P}(v \in G(n) \text{ is chosen}) = \frac{D_{out}(v) + \mu}{n + \mu N(n)}.$$

For $i, j = 0, 1, 2, \dots$ and $n \geq n_0$, let $N_{ij}(n)$ be the random number of nodes in $G(n)$ with in-degree i and out-degree j . According to [4, Theorem 3.2], there exist non-random constants $p(i, j)$ such that

$$\lim_{n \rightarrow \infty} \frac{N_{ij}(n)}{N(n)} = p(i, j) \quad \text{a.s. for } i, j = 0, 1, 2, \dots \quad (2.5)$$

Define two random variables (I, O) such that

$$P[I = i, O = j] = p(i, j), \quad i, j = 0, 1, 2, \dots$$

and the distribution generated by (I, O) is a non-standard regularly varying measure [25, 26]. The pair (I, O) has representation

$$(I, O) \stackrel{d}{=} B(1 + X_1, Y_1) + (1 - B)(X_2, 1 + Y_2), \quad (2.6)$$

where B is a Bernoulli switching variable independent of $X_j, Y_j, j = 1, 2$ with

$$\mathbf{P}(B = 1) = 1 - \mathbf{P}(B = 0) = \frac{\gamma}{\alpha + \gamma}.$$

Let $T_\delta(p)$ be a negative binomial integer valued random variable with parameters $\delta > 0$ and $p \in (0, 1)$. Now suppose $\{T_{\delta_1}(p), p \in (0, 1)\}$ and $\{\tilde{T}_{\delta_2}(p), p \in (0, 1)\}$ are two independent families of negative binomial random variables and define

$$c_1 = \frac{\alpha + \beta}{1 + \lambda(\alpha + \gamma)}, \quad c_2 = \frac{\beta + \gamma}{1 + \mu(\alpha + \gamma)} \quad \text{and } a = c_2/c_1.$$

By [26, Theorem 2], $X_j, Y_j, j = 1, 2$ in (2.6) can be written as

$$(X_1, Y_1) = (T_{\lambda+1}(Z^{-1}), \tilde{T}_\mu(Z^{-a})), \text{ and } (X_2, Y_2) = (T_\lambda(Z^{-1}), \tilde{T}_{\mu+1}(Z^{-a})), \quad (2.7)$$

where Z is a Pareto random variable on $[1, \infty)$ with index c_1^{-1} , independent of the negative binomial random variables.

We will show that $p(i, j)$ in (2.5) is regularly varying.

3 Embeddability and Monotonicity.

The embeddability problem is no longer as straightforward in higher dimensions as Theorem 2.1 would lead us to believe; here we only deal with the case where $d = 2$. Multivariate regular variation provides no control over the function's variation when we move from ray to ray and in order to obtain embeddability in the bivariate case we need regularity conditions. The following theorem provides one approach.

Theorem 3.1 (Standard case). *Suppose $u : \mathbb{Z}_+^2 \setminus \{\mathbf{0}\} \mapsto \mathbb{R}_+$ is an eventually decreasing (in both arguments) mass function that is regularly varying: There exists $h(\cdot) \in RV_\rho$, $\rho < 0$ such that,*

$$\lim_{n \rightarrow \infty} \frac{u([nx], [ny])}{h(n)} = \lambda(x, y) > 0, \quad \forall x, y > 0. \quad (3.1)$$

Then the function $g(x, y) := u([x], [y])$ is eventually decreasing and regularly varying,

$$\lim_{t \rightarrow \infty} \frac{g(tx, ty)}{h(t)} = \lambda(x, y), \quad \forall x, y > 0, \quad (3.2)$$

so that u is embeddable in the regularly varying function g .

Remark 3.1. In particular, if a probability mass function (pmf) $p(i, j)$ plays the role of u and satisfies monotonicity and (3.1), then we can embed $p(i, j)$ in a monotone regularly varying probability density function (pdf) $f(x, y)$ by simply assigning constant probability density $p([x], [y])$ over the rectangle $[x, [x + 1]] \times [y, [y] + 1]$.

Proof. For $s > 0$, we show that the limiting function λ in (3.1) satisfies $\lambda(sx, sy) = s^\rho \lambda(x, y)$ as is the case for any function satisfying (3.1) and embeddability. For fixed $s > 0$, (3.1) gives

$$\lim_{n \rightarrow \infty} \frac{u([nsx], [nsy])}{h(n)} = \lambda(sx, sy).$$

We rewrite the left hand side as

$$\frac{u([nsx], [nsy])}{h(n)} = \frac{u([nsx], [nsy])}{u([ns]x, [ns]y)} \times \frac{u([ns]x, [ns]y)}{h([ns])} \times \frac{h([ns])}{h(n)}. \quad (3.3)$$

The second term on the right hand side of (3.3) converges to $\lambda(x, y)$ as $n \rightarrow \infty$, by (3.1), and as $h \in RV_\rho$, the third term converges to s^ρ . It now suffices to show

$$\lim_{n \rightarrow \infty} \frac{u([nsx], [nsy])}{u([ns]x, [ns]y)} = 1. \quad (3.4)$$

Since u is eventually decreasing, for $x > 0, y > 0$ and n large enough $u([nx], [ny])$ is decreasing in both x and y . Hence,

$$\frac{u([([ns] + 1)x], [([ns] + 1)y])}{u([ns]x, [ns]y)} \leq \frac{u([nsx], [nsy])}{u([ns]x, [ns]y)} \leq 1, \quad (3.5)$$

and

$$\begin{aligned} \frac{u([(ns] + 1)x], [(ns] + 1)y)}{u([ns]x, [ns]y)} &= \frac{u([(ns] + 1)x], [(ns] + 1)y)}{h([ns] + 1)} \frac{h([ns] + 1)}{h([ns])} \frac{h([ns])}{u([ns]x, [ns]y)} \\ &\xrightarrow{n \rightarrow \infty} \lambda(x, y) \cdot 1 \cdot \frac{1}{\lambda(x, y)} = 1, \end{aligned} \quad (3.6)$$

which coupled with (3.5) shows (3.4). Set $g(x, y) = u([x], [y])$ and on the one hand, as $t \rightarrow \infty$,

$$\begin{aligned} \frac{g(tx, ty)}{h(t)} &\sim \frac{u([tx], [ty])}{h([t])} \leq \frac{u([t]x, [t]y)}{h([t])} \\ &\rightarrow \lambda(x, y), \end{aligned}$$

and on the other, since $([t] + 1)x \geq tx$,

$$\begin{aligned} \frac{g(tx, ty)}{h(t)} &\geq \frac{g([t] + 1)x, ([t] + 1)y)}{h(t)} \\ &= \frac{u([t] + 1)x, [t] + 1)y)}{h(t)}, \end{aligned}$$

and the result follows from (3.6). \square

We generalize Theorem 3.1 to the non-standard case.

Corollary 3.1 (Non-standard case). *Suppose $u : \mathbb{Z}_+^2 \setminus \{\mathbf{0}\} \mapsto \mathbb{R}_+$ is a mass function that is eventually decreasing in both arguments. For $\alpha_1, \alpha_2 > 0$, let $b_1 \in RV_{1/\alpha_1}$ and $b_2 \in RV_{1/\alpha_2}$ be two strictly increasing scaling functions such that*

$$\lim_{n \rightarrow \infty} \frac{u([b_1(n)x], [b_2(n)y])}{h(n)} = \lambda(x, y) > 0, \quad \forall x, y > 0, \quad (3.7)$$

where $h(\cdot) \in RV_\rho$, $\rho < 0$. The limit function λ satisfies the scaling property: $\lambda(s^{1/\alpha_1}x, s^{1/\alpha_2}y) = s^\rho \lambda(x, y)$ for all $s, x, y > 0$ and $g(x, y) = u([x], [y])$ is an eventually decreasing regularly varying function such that and for all $x, y > 0$

$$\lim_{t \rightarrow \infty} \frac{g(b_1(t)x, b_2(t)y)}{h(t)} = \lambda(x, y) > 0. \quad (3.8)$$

Thus u is embeddable in the regularly varying function g .

Proof. By [19, Theorem 1.2.2], (3.7) implies that for all $x, y > 0$,

$$\lim_{n \rightarrow \infty} \frac{u([b_1(nx)], [b_2(ny)])}{h(n)} = \lambda(x^{1/\alpha_1}, y^{1/\alpha_2}).$$

Since b_1 and b_2 are strictly increasing, $u([b_1(x)], [b_2(y)])$ is eventually decreasing in x and y . A similar proof as in Theorem 3.1 shows that the function g defined by $g(b_1(x), b_2(y)) := u([b_1(x)], [b_2(y)])$ satisfies for all $x, y > 0$,

$$\lim_{t \rightarrow \infty} \frac{g(b_1(tx), b_2(ty))}{h(t)} = \lambda(x^{1/\alpha_1}, y^{1/\alpha_2}).$$

Then the scaling property of λ follows immediately. Applying [19, Theorem 1.2.2] again, we conclude that u can be embedded in a function g satisfying (3.8). \square

Assuming a probability measure has a mass function satisfying the monotonicity condition, we make explicit the relation between regular variation of the probability measure and regular variation of its pmf. The first part of Theorem 3.2 is the multivariate analogue of the *monotone density theorem*.

Theorem 3.2. *Suppose two non-negative integer valued random variables (X, Y) have mass function $p(i, j)$ and p is eventually decreasing in both arguments. For $\alpha_1, \alpha_2 > 0$, assume further that $b_1(\cdot) \in RV_{1/\alpha_1}$, $b_2(\cdot) \in RV_{1/\alpha_2}$ are strictly increasing, continuous scaling functions.*

1. (REGULAR VARIATION OF THE MEASURE IMPLIES REGULAR VARIATION OF THE MASS FUNCTION:) *If there exists a limit measure $\nu \in \mathbb{M}(\mathbb{R}_+^2 \setminus \{\mathbf{0}\})$ with density $\lambda(x, y)$, such that as $t \rightarrow \infty$,*

$$t\mathbf{P}\left[\left(\frac{X}{b_1(t)}, \frac{Y}{b_2(t)}\right) \in \cdot\right] \rightarrow \nu(\cdot), \quad \text{in } \mathbb{M}(\mathbb{R}_+^2 \setminus \{\mathbf{0}\}), \quad (3.9)$$

then the mass function $p(i, j)$ is regularly varying:

$$\lim_{t \rightarrow \infty} \frac{p([b_1(t)x], [b_2(t)y])}{(tb_1(t)b_2(t))^{-1}} = \lambda(x, y), \quad x, y > 0. \quad (3.10)$$

2. (REGULAR VARIATION OF THE MASS FUNCTION IMPLIES REGULAR VARIATION OF THE MEASURE:) *Conversely, if $p(i, j)$ satisfies (3.10), then the distribution of (X, Y) is regularly varying and (3.9) holds with $\nu(dx, dy) = \lambda(x, y)dx dy$.*

Proof. (1) Extend the pmf $p(i, j)$ to an eventually decreasing pdf $f(x, y)$ using the method in Remark 3.1, such that $p(i, j) = f(i, j)$ for all $i, j \in \mathbb{N}$. Let (X^*, Y^*) have pdf $f(x, y)$, and (3.9) implies

$$t\mathbf{P}\left[\left(\frac{X^*}{b_1(t)}, \frac{Y^*}{b_2(t)}\right) \in \cdot\right] \rightarrow \nu(\cdot), \quad \text{as } t \rightarrow \infty.$$

By [7, Theorem 2.3], we have

$$\lim_{t \rightarrow \infty} \frac{f(b_1(t)x, b_2(t)y)}{(tb_1(t)b_2(t))^{-1}} = \lambda(x, y).$$

Then (3.10) can be recovered by noting that $p([b_1(t)x], [b_2(t)y]) = f(b_1(t)x, b_2(t)y)$ and $p(i, j)$ is embeddable in $f(x, y)$ in the sense of (3.2).

(2) Since $p(i, j)$ satisfies (3.10) and is eventually monotone, $p(i, j)$ can be embedded into a pdf, $f(x, y)$ which is also eventually monotone. Following the arguments in [8, 7], we have for all Borel sets $A \subseteq \mathbb{R}_+^2 \setminus \{\mathbf{0}\}$ bounded away from $\mathbf{0}$ such that $\int_A \lambda(x, y)dx dy < \infty$,

$$\lim_{t \rightarrow \infty} \int_A \frac{f(b_1(t)x, b_2(t)y)}{(tb_1(t)b_2(t))^{-1}} dx dy = \int_A \lambda(x, y) dx dy,$$

and (3.9) then follows by the embeddability of $p(i, j)$. □

3.1 Application to the Preferential Attachment Model

As an example of the previous results, we treat a special case of the preferential attachment model, where $c_1 = c_2$, i.e. the joint regular variation of the distribution of (I, O) is standard. Due to the decomposition in (2.6), it suffices to verify the monotonicity of the joint mass function of (X_j, Y_j) , $j = 1, 2$. We will show that the joint mass function of (X_1, Y_1) is decreasing and the same argument

also works for (X_2, Y_2) . Our approach requires the fact that the gamma function $\Gamma(x)$ is log-convex and $(\log \Gamma)'' > 0$. See [1, page 260, Section 6.4]. This means $f(x) = \log \Gamma(x)$ is convex, $f'(x) = \Gamma'(x)/\Gamma(x)$ and $f''(x) = (\Gamma''(x)\Gamma(x) - (\Gamma'(x))^2)/\Gamma(x)^2 > 0$.

Recall Section 2.3 and (2.7) and (2.6). When $c_1 = c_2$, we have $(X_1, Y_1) = (T_{\lambda+1}(Z^{-1}), \tilde{T}_\mu(Z^{-1}))$. Set $q(i, j) = \mathbf{P}[X_1 = i, Y_1 = j]$ and for $i, j \in \mathbb{N}$,

$$\begin{aligned} q(i, j) &:= \mathbf{P}[T_{\lambda+1}(Z^{-1}) = i, \tilde{T}_\mu(Z^{-1}) = j] \\ &= \frac{\Gamma(i + \lambda + 1)}{\Gamma(\lambda + 1)\Gamma(i + 1)} \frac{\Gamma(j + \mu)}{\Gamma(\mu)\Gamma(j + 1)} \cdot \frac{1}{c_1} \int_1^\infty z^{-(2+1/c_1+\lambda+\mu)} (1 - z^{-1})^{i+j} dz \\ &= \left(\frac{\Gamma(1 + 1/c_1 + \lambda + \mu)}{c_1 \Gamma(\lambda + 1)\Gamma(\mu)} \right) \times \frac{\Gamma(i + \lambda + 1)/\Gamma(i + 1) \cdot \Gamma(j + \mu)/\Gamma(j + 1)}{\Gamma(i + j + 1/c_1 + \lambda + \mu + 2)/\Gamma(i + j + 1)} \\ &=: C \frac{\Gamma(i + \lambda + 1)/\Gamma(i + 1) \cdot \Gamma(j + \mu)/\Gamma(j + 1)}{\Gamma(i + j + 1/c_1 + \lambda + \mu + 2)/\Gamma(i + j + 1)}. \end{aligned} \quad (3.11)$$

Taking the log of $q(i, j)$, pretend i is a continuous variable and take the first partial derivative with respect to i . This gives

$$\begin{aligned} \frac{\partial}{\partial i} \log q(i, j) &= \left[\frac{\Gamma'(i + \lambda + 1)}{\Gamma(i + \lambda + 1)} - \frac{\Gamma'(i + 1)}{\Gamma(i + 1)} \right] - \left[\frac{\Gamma'(i + j + 1/c_1 + \lambda + \mu + 2)}{\Gamma(i + j + 1/c_1 + \lambda + \mu + 2)} - \frac{\Gamma'(i + j + 1)}{\Gamma(i + j + 1)} \right] \\ &= [f'(i + \lambda + 1) - f'(i + 1)] - [f'(i + j + 1/c_1 + \lambda + \mu + 2) - f'(i + j + 1)], \end{aligned}$$

where we used the notation $f = \log \Gamma$. Since f is convex, f' is increasing on \mathbb{R}_+ . For fixed $x > 0$ and $a > 0$, $g(a) := f'(x + a) - f'(x)$, $a > 0$, satisfies $g'(a) = f''(x + a) > 0$, and thus $g(a)$ is increasing in a . Hence for $j \geq 0$,

$$f'(i + j + 1/c_1 + \lambda + \mu + 2) - f'(i + j + 1) \geq f'(i + 1/c_1 + \lambda + \mu + 2) - f'(i + 1),$$

which gives

$$\begin{aligned} \frac{\partial}{\partial i} \log q(i, j) &\leq [f'(i + \lambda + 1) - f'(i + 1)] - [f'(i + 1/c_1 + \lambda + \mu + 2) - f'(i + 1)] \\ &= f'(i + \lambda + 1) - f'(i + 1/c_1 + \lambda + \mu + 2) < 0. \end{aligned}$$

Therefore, $q(i, j)$, the joint mass function of (X_1, Y_1) , is decreasing in i . Analogously we can show that $q(i, j)$ is also decreasing in j . The monotonicity of the joint mass function $p(i, j)$ of (I, O) defined in (2.5) then follows from the decomposition in (2.6). Hence we are left to show (3.1) for $p(i, j)$.

Using (3.11) with the constant C restored, we have

$$q([nx], [ny]) = \left(\frac{\Gamma(1 + 1/c_1 + \lambda + \mu)}{c_1 \Gamma(\lambda + 1)\Gamma(\mu)} \right) \times \frac{\Gamma([nx] + \lambda + 1)/\Gamma([nx] + 1) \cdot \Gamma([ny] + \mu)/\Gamma([ny] + 1)}{\Gamma([nx] + [ny] + 1/c_1 + \lambda + \mu + 2)/\Gamma([nx] + [ny] + 1)}.$$

Hence, for all $x, y > 0$

$$\lim_{n \rightarrow \infty} \frac{q([nx], [ny])}{n^{-(2+1/c_1)}} = \frac{\Gamma(1 + 1/c_1 + \lambda + \mu)}{c_1 \Gamma(\lambda + 1)\Gamma(\mu)} \frac{x^\lambda y^{\mu-1}}{(x + y)^{1+\lambda+\mu+1/c_1}}, \quad (3.12)$$

which shows that $q(i, j)$ is regularly varying. Similar calculations can be done for (X_2, Y_2) and then using (2.6), we have for all $x, y > 0$,

$$\lim_{n \rightarrow \infty} \frac{p([nx], [ny])}{n^{-(2+1/c_1)}} = \frac{\gamma}{\alpha + \gamma} \frac{\Gamma(1 + 1/c_1 + \lambda + \mu)}{c_1 \Gamma(\lambda + 1)\Gamma(\mu)} \frac{x^\lambda y^{\mu-1}}{(x + y)^{1+\lambda+\mu+1/c_1}}$$

$$+ \frac{\alpha}{\alpha + \gamma} \frac{\Gamma(1 + 1/c_1 + \lambda + \mu)}{c_1 \Gamma(\lambda) \Gamma(\mu + 1)} \frac{x^{\lambda-1} y^\mu}{(x + y)^{1+\lambda+\mu+1/c_1}}. \quad (3.13)$$

Therefore, in the special case that $c_1 = c_2$, $p(i, j)$ is embeddable and standard regularly varying.

We also know from [26, Theorem 2] that the joint distribution (as opposed to the mass function) of (I, O) is regularly varying, and Theorem 3.2 implies that the joint mass function $p(i, j)$ must also be regularly varying in the sense of (3.10). From (3.13), the limit function for $p(i, j)$ is exactly the density for the limit measure specified in [26]. We summarize:

Proposition 3.1. *In the preferential attachment model with $c_1 = c_2$, the asymptotic joint mass function of in- and out-degrees $p(i, j)$ is eventually monotone, embeddable and standard regularly varying and satisfies (3.13).*

When $c_1 \neq c_2$, we have not succeeded in demonstrating the monotonicity condition for the mass function. Instead, in the next section, we will give a different sufficient condition which can be used to verify the regular variation of the joint mass function.

4 Embeddability and Convergence on the Unit Sphere.

In Section 3, embeddability of a mass function is guaranteed by assuming monotonicity; however, sometimes monotonicity is either not applicable or difficult to verify. An alternate approach is to fix a norm $\|\cdot\|$ and suppose regular variation on

$$\aleph_0 := \{\mathbf{v} \in \mathbb{R}_+^2 : \|\mathbf{v}\| = 1\},$$

the unit sphere in this norm relative to the origin, with respect to a continuous variable. This is sufficient for embeddability but further uniformity and boundedness conditions are necessary to relate pmf's, pdf's and their measures.

Theorem 4.1 (Standard Case). *Suppose $h(\cdot) \in RV_\rho$, $\rho < 0$ and $u : \mathbb{Z}_+^2 \mapsto \mathbb{R}_+$ satisfies: there exists a limit function $\lambda_0 > 0$ defined on \aleph_0 such that*

$$\lim_{t \rightarrow \infty} \frac{u([tx], [ty])}{h(t)} = \lambda_0(x, y), \quad \forall (x, y) \in \aleph_0. \quad (4.1)$$

Then

1. The doubly indexed function $u(i, j)$ is regularly varying: For all $x, y > 0$

$$\lim_{n \rightarrow \infty} \frac{u([nx], [ny])}{h(n)} = \lambda(x, y) := \lambda_0\left(\frac{(x, y)}{\|(x, y)\|}\right) \|(x, y)\|^\rho > 0; \quad (4.2)$$

2. The doubly indexed function $u(i, j)$ is embeddable in a regularly varying function $f : \mathbb{R}_+^2 \mapsto \mathbb{R}$ with limit function $\lambda(\cdot)$ such that $f(x, y) = u([x], [y])$;
3. If (4.1) is uniform on \aleph_0 , then f also satisfies

$$\lim_{t \rightarrow \infty} \sup_{(x, y) \in \aleph_0} \left| \frac{f(tx, ty)}{h(t)} - \lambda_0(x, y) \right| = 0. \quad (4.3)$$

Note if a pdf satisfies (4.3) with λ_0 positive and bounded on \aleph_0 , then also the measure is regularly varying. See [9, 24].

Proof. Write $\mathbf{v} := (x, y)$, $u([t\mathbf{v}]) := u([tx], [ty])$, and for \mathbf{v} such that $\|\mathbf{v}\| \neq 0$, we have $\mathbf{a} := \mathbf{v}/\|\mathbf{v}\| \in \mathfrak{N}_0$. We show that convergence on \mathfrak{N}_0 implies convergence everywhere. We have for $\mathbf{v} = \|\mathbf{v}\| \cdot \mathbf{a} \neq \mathbf{0}$,

$$\frac{u([t\mathbf{v}])}{h(t)} = \frac{u([t\|\mathbf{v}\| \cdot \mathbf{a}])}{h(t)} = \frac{u([t\|\mathbf{v}\| \cdot \mathbf{a}])}{h(t\|\mathbf{v}\|)} \cdot \frac{h(t\|\mathbf{v}\|)}{h(t)} \rightarrow \lambda_0(\mathbf{a})\|\mathbf{v}\|^\rho.$$

This gives (4.2) and regular variation of f follows directly. If (4.1) is uniform on \mathfrak{N}_0 then by definition of $f(\mathbf{v}) = u([\mathbf{v}])$, (4.3) is true. \square

There are various possible ways to extend this result to the non-standard case, depending on the purpose in mind. The following is crafted with Section 4.1 in mind.

Corollary 4.1 (Non-standard case; power law scaling). *Suppose $h(\cdot) \in RV_\rho$, $\rho < 0$, $u : \mathbb{Z}_+^2 \mapsto \mathbb{R}_+$ and scaling functions are power laws; i.e., $b_i(t) = t^{1/\alpha_i}$, $i = 1, 2$. If there exists a limit function $\lambda_0 > 0$ defined on*

$$\mathcal{E}_0 := \{(x, y) : (x^{\alpha_1}, y^{\alpha_2}) \in \mathfrak{N}_0\}$$

such that

$$\lim_{t \rightarrow \infty} \frac{u([t^{1/\alpha_1}x], [t^{1/\alpha_2}y])}{h(t)} = \lambda_0(x, y), \quad \forall (x, y) \in \mathcal{E}_0, \quad (4.4)$$

then

1. *The doubly indexed function $u(i, j)$ is regularly varying: For all $x, y > 0$, define $\mathbf{w} = \mathbf{w}(x, y) := (x^{\alpha_1}, y^{\alpha_2})$ and*

$$\lim_{n \rightarrow \infty} \frac{u([n^{1/\alpha_1}x], [n^{1/\alpha_2}y])}{h(n)} = \lambda(x, y) := \lambda_0\left(\frac{x}{\|\mathbf{w}\|^{1/\alpha_1}}, \frac{y}{\|\mathbf{w}\|^{1/\alpha_2}}\right) \|\mathbf{w}\|^\rho > 0; \quad (4.5)$$

2. *The doubly indexed function $u(i, j)$ is embeddable in a non-standard regularly varying function $f : \mathbb{R}_+^2 \mapsto \mathbb{R}$ with limit function $\lambda(\cdot)$ such that $f(x, y) = u([x], [y])$;*
3. *If convergence in (4.4) is uniform on \mathcal{E}_0 , then also,*

$$\lim_{t \rightarrow \infty} \sup_{(x, y) \in \mathcal{E}_0} \left| \frac{f(t^{1/\alpha_1}x, t^{1/\alpha_2}y)}{h(t)} - \lambda_0(x, y) \right| = 0. \quad (4.6)$$

Remark 4.1. If a regularly varying pdf $f(x, y)$ on \mathbb{R}_+^2 satisfies (4.6) with λ_0 positive and bounded on \mathcal{E}_0 , then also the measure induced by the pdf is regularly varying. If a pmf $u(i, j)$ satisfies (4.4) with λ positive and bounded on \mathcal{E}_0 , then the corresponding discretely supported measure is regularly varying. See [9, 24] and the comments after the proof of Corollary 4.1.

Proof. Note that for all $(x, y) \neq \mathbf{0}$, the function \mathbf{w} creates a map onto \mathcal{E}_0 and

$$(x, y) \mapsto \left(\frac{x}{\|\mathbf{w}\|^{1/\alpha_1}}, \frac{y}{\|\mathbf{w}\|^{1/\alpha_2}} \right) \in \mathcal{E}_0.$$

We show that convergence on \mathcal{E}_0 implies convergence for all $x, y > 0$: For $x, y > 0$,

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{u([t^{1/\alpha_1}x], [t^{1/\alpha_2}y])}{h(t)} &= \lim_{t \rightarrow \infty} \frac{u([t^{1/\alpha_1}\|\mathbf{w}\|^{1/\alpha_1} \cdot (x/\|\mathbf{w}\|^{1/\alpha_1})], [t^{1/\alpha_2}\|\mathbf{w}\|^{1/\alpha_2} \cdot (y/\|\mathbf{w}\|^{1/\alpha_2})])}{h(t)} \\ &= \lim_{t \rightarrow \infty} \frac{u([(t\|\mathbf{w}\|)^{1/\alpha_1}x/\|\mathbf{w}\|^{1/\alpha_1}], [(t\|\mathbf{w}\|)^{1/\alpha_2}y/\|\mathbf{w}\|^{1/\alpha_2}])}{h(t\|\mathbf{w}\|)} \cdot \frac{h(t\|\mathbf{w}\|)}{h(t)} \end{aligned}$$

$$= \lambda_0 \left(\frac{x}{\|\mathbf{w}\|^{1/\alpha_1}}, \frac{y}{\|\mathbf{w}\|^{1/\alpha_2}} \right) \|\mathbf{w}\|^\rho,$$

which verifies (4.5). The embeddability and (4.6) follow by a similar argument as in the proof of Theorem 4.1. \square

For Remark 4.1: If we assume f is regularly varying, λ_0 is bounded on \mathcal{E}_0 and (4.6), it is straightforward to generalize results in [9, Theorem 2.1]: For $(x, y) \in A$, where A is a Borel set bounded away from $\mathbf{0}$, find an integrable bound for

$$\frac{f((t\|\mathbf{w}\|)^{1/\alpha_1} \|\mathbf{w}\|^{-1/\alpha_1} x, (t\|\mathbf{w}\|)^{1/\alpha_2} \|\mathbf{w}\|^{-1/\alpha_2} y)}{h(t\|\mathbf{w}\|)} \cdot \frac{h(t\|\mathbf{w}\|)}{h(t)},$$

using the boundedness of λ_0 on \mathcal{E}_0 to bound the first term and Karamata's representation to bound the second (same as in the proof of [9, Theorem 2.1]). Then the convergence of the associated measure follows from dominated convergence.

4.1 Application to the Preferential Attachment Model.

We now apply Corollary 4.1 to show that in the case where $c_1 \neq c_2$, the joint pmf of (I, O) is also regularly varying. The following lemma is a variant of Stirling's formula.

Lemma 4.1. *For a compact set $K \subset (0, \infty)$ and $0 < k \neq 1$, we have*

$$\lim_{t \rightarrow \infty} \sup_{x \in K} \left| \frac{\Gamma(tx + k)}{t^k \Gamma(tx)} - x^k \right| = 0.$$

Proof. Define $f_t(x) = \frac{\Gamma(tx+k)}{t^k \Gamma(tx)}$. By Stirling's formula [1, p. 254], we have pointwise convergence:

$$\lim_{t \rightarrow \infty} f_t(x) = x^k, \quad x > 0. \quad (4.7)$$

According to [21, Theorem 4] or [22, Theorem 1.3], for $x > 0$, $\log f_t(x)$ is increasing in t if $k \in (0, 1)$ and decreasing in t if $k > 1$. Either case will allow us to apply Dini's theorem to the pointwise convergence in (4.7) to conclude the uniform convergence of f_t on K . \square

Now we show regular variation of $p(i, j)$ for the preferential attachment model in the nonstandard case where $c_1 \neq c_2$. Here we still only detail the calculations for (X_1, Y_1) ; results for (X_2, Y_2) can be obtained in a similar way.

Set $a = c_2/c_1$, $b_i(t) = t^{c_i}$. We have

$$\begin{aligned} q([t^{c_1}x], [t^{c_2}y]) &= \frac{\Gamma([t^{c_1}x] + \lambda + 1)}{\Gamma(\lambda + 1)\Gamma([t^{c_1}x] + 1)} \frac{\Gamma([t^{c_2}y] + \mu)}{\Gamma(\mu)\Gamma([t^{c_2}y] + 1)} \frac{1}{c_1} \times \\ &\times \int_1^\infty s^{-(1+1/c_1)} (1-s^{-1})^{[t^{c_1}x]} s^{-(\lambda+1)} (1-s^{-a})^{[t^{c_2}y]} s^{-a\mu} ds. \end{aligned}$$

It suffices to assume according to Corollary 4.1 that $(x, y) \in \mathcal{E}_0$. Making the change of variable $z = s/t^{c_1}$, we rewrite the integral as

$$t^{-(1+c_1(\lambda+1)+c_2\mu)} \int_{1/t^{c_1}}^\infty z^{-(2+1/c_1+\lambda+a\mu)} \left(1 - \frac{z^{-1}}{t^{c_1}}\right)^{[t^{c_1}x]} \left(1 - \frac{z^{-a}}{t^{c_2}}\right)^{[t^{c_2}y]} dz. \quad (4.8)$$

Note that for any (x, y) on the compact set \mathcal{E}_0 defined in Corollary 4.1, dominated convergence gives the convergence of the integral in (4.8) to

$$\int_0^\infty z^{-(2+1/c_1+\lambda+a\mu)} e^{-(\frac{x}{z}+\frac{y}{z^a})} dz,$$

as $t \rightarrow \infty$. Therefore, according to Corollary 4.1 we have for all $x, y > 0$,

$$\lim_{n \rightarrow \infty} \frac{q([n^{c_1}x], [n^{c_2}y])}{n^{-(1+c_1+c_2)}} = \frac{x^\lambda y^{\mu-1}}{c_1 \Gamma(\lambda+1) \Gamma(\mu)} \int_0^\infty z^{-(2+1/c_1+\lambda+a\mu)} e^{-(\frac{x}{z}+\frac{y}{z^a})} dz, \quad (4.9)$$

which shows that $q(i, j)$ is regularly varying. Applying (2.6) again, we conclude,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{p([n^{c_1}x], [n^{c_2}y])}{n^{-(1+c_1+c_2)}} &= \frac{\gamma}{\alpha + \gamma} \frac{x^\lambda y^{\mu-1}}{c_1 \Gamma(\lambda+1) \Gamma(\mu)} \int_0^\infty z^{-(2+1/c_1+\lambda+a\mu)} e^{-(\frac{x}{z}+\frac{y}{z^a})} dz \\ &\quad + \frac{\alpha}{\alpha + \gamma} \frac{x^{\lambda-1} y^\mu}{c_1 \Gamma(\lambda) \Gamma(\mu+1)} \int_0^\infty z^{-(1+a+1/c_1+\lambda+a\mu)} e^{-(\frac{x}{z}+\frac{y}{z^a})} dz. \end{aligned} \quad (4.10)$$

Therefore, $p(i, j)$ is regularly varying with scaling functions $b_i(t) = t^{c_i}$ for $i = 1, 2$ and limit function as in (4.10), which is the density of the limit measure given in [26]. We summarize:

Proposition 4.1. *In a preferential attachment model where $c_1 \neq c_2$, the joint mass function of in- and out-degrees $p(i, j)$ is embeddable, nonstandard regularly varying and satisfies (4.10).*

Using Lemma 4.1, we can also conclude that the convergence in (4.9) is uniform on \mathcal{E}_0 and so is (4.10). The limit function specified in (4.10) is positive and bounded on \mathcal{E}_0 . Hence, the results in Corollary 4.1(3) and Remark 4.1 are also applicable here and allow the conclusion that regular variation of the pmf $p(i, j)$ implies regular variation of the associated measure proven in [25, 26]

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